Technical Notes for the Raytracing of Toroidal Surfaces

Table of Contents

Johannes Störkle, 2020-03-18

Introduction

In geometry, a torus is a surface of revolution generated by revolving a circle in three-dimensional space about an axis that is coplanar with the circle.
In optics, a toroidal surface is described with the circle radius r_x and the distance R_y to the y-axis, known as the radius of rotation.
According to ZEMAX, the curve in the Y-Z plane is defined by

$$z_{x=0} = \frac{\frac{1}{r_x} y^2}{1 + \sqrt{1 - (1+\kappa)\frac{1}{r_x^2} y^2}} + \alpha_1 y^2 + \alpha_2 y^4 + \cdots, \qquad (1)$$

where κ is the conic constant with respect to the y-direction. However, the complete three-dimensional formulation is not given in [Zemax11][1]. The toroidal surface is shown in Figure 1, whereby the blue curve indicates the curve given by equation (1).

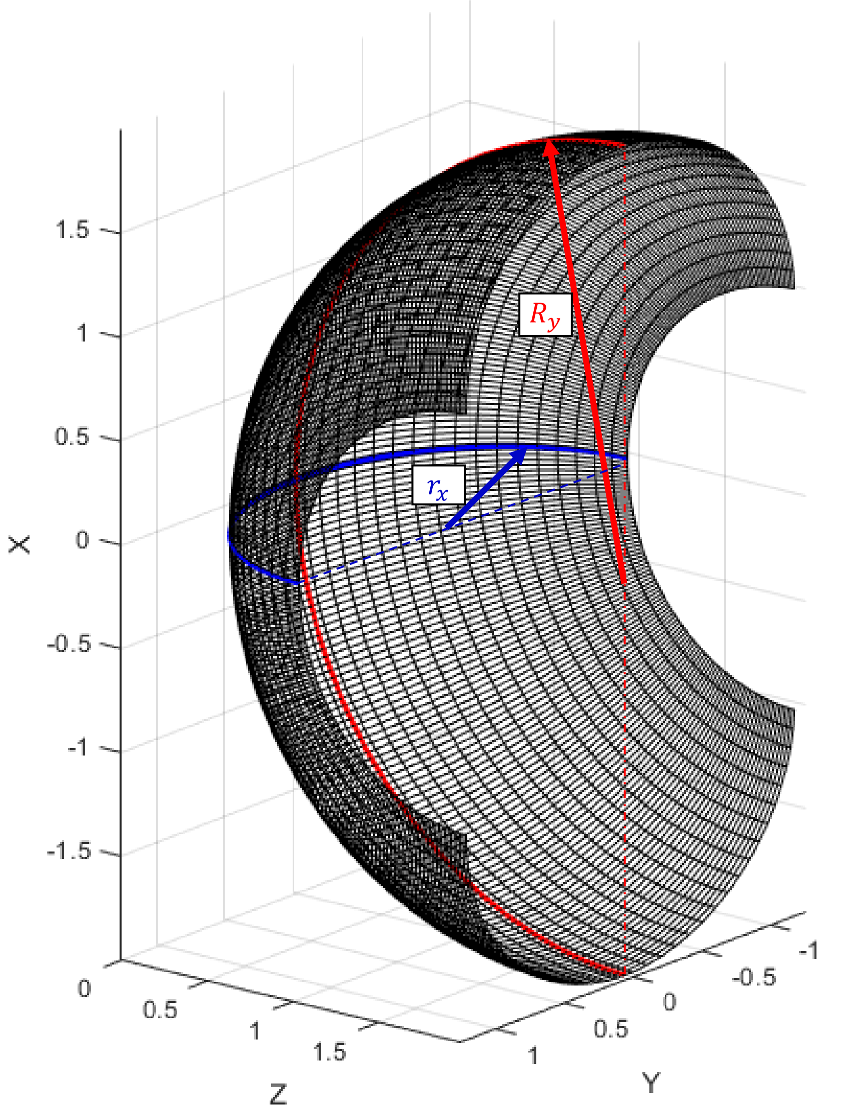

Figure 1: Toroidal Surface with the circle radius r_x and the radius of rotation R_y

[1] [Zemax11] Zemax: Zemax Optical Design Program - User's Manual. Radiant ZEMAX LLC, 2011.

Toroidal Surface Description

If the radii in the range of $0 \leq r_x \leq R_y$, the toroidal surface in three-dimensional space **without a conic constant** can be described with

$$z = R_y - \sqrt{\left(R_y - r_x + \sqrt{r_x^2 - y^2}\right)^2 - x^2}$$

as given in [Gross05][2]. In order to derive the three-dimensional formulation with a conic constant, one can first regard the curve within the y-z plane ($x = 0$) it yields

$$z = R_y - \left(R_y - r_x + \sqrt{r_x^2 - y^2}\right) = r_x - \sqrt{r_x^2 - y^2}$$

The last formulation is similar to the description of a spherical surface $z_{\text{Sphere}} = R_i - \sqrt{R_i^2 - \rho^2}$.

Furthermore, the introduction of the conic constant leads to

$$z_{\text{conicSphere}} = \frac{1}{1+\kappa}\left(R_i - \sqrt{R_i^2 - (1+\kappa)\rho^2}\right) = \frac{\rho^2}{R_i + \sqrt{R_i^2 - (1+\kappa)\rho^2}} = \frac{\frac{1}{R_i}\rho^2}{1 + \sqrt{1 - (1+\kappa)\frac{1}{R_i^2}\rho^2}}$$

as derived in [Stoerkle18][3]. Hereby, ρ is the radial coordinate and a binominal expansion is used for the arrangement. The last result has the same structure as the first term in equation (1). Thus, the formulation of the conic toroidal surface results in analogy

$$z = R_y - \sqrt{\left(R_y + \frac{1}{1+\kappa}\left(-r_x + \sqrt{r_x^2 - (1+\kappa)y^2}\right)\right)^2 - x^2}.$$

This can be generalized to values of $r_x \in \mathbb{R}$ and $R_y \in \mathbb{R}$ with

$$z = R_y - \operatorname{sign}(r_x)\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - r_x\right)\right)^2 - x^2}, \tag{2}$$

$$\text{if } |R_y| \geq \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x \geq 0\text{, or } |R_y| \leq \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x \leq 0$$

and

$$z = R_y - \operatorname{sign}(r_x)\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(-\sqrt{r_x^2 - (1+\kappa)y^2} - r_x\right)\right)^2 - x^2} \tag{3}$$

$$\text{if } |R_y| < \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x > 0\text{, or } |R_y| > \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x < 0.$$

In order to keep the vertex at the origin, i.e. $z(0,0) = 0$, a constant z-shift must be applied. This can be formulated for the following cases,

If $R_y \geq \frac{r_x}{1+\kappa}$

If $R_y r_x \geq 0$

$$z_{\text{corr}} = z \tag{4}$$

[2] [Gross05] Gross, H. (Ed.): Handbook of Optical Systems Vol. 1 - Fundamentals of Technical Optics. Weinheim: Wiley-VCH Verlag, 2005. (see also PPT IAP Jena)

[3] [Stoerkle18] Störkle, J.: Dynamic Simulation and Control of Optical Systems. No. 58 in Dissertation, Schriften aus dem Institut für Technische und Numerische Mechanik der Universität Stuttgart. Shaker Verlag, Aachen (2018). Shaker Verlag

If $R_y r_x < 0$

$$z_{\text{corr}} = z - 2R \tag{5}$$

If $R_y < \frac{r_x}{1+\kappa}$

If $R_y r_x \geq 0$

$$z_{\text{corr}} = z + 2\left(\frac{r_x}{1+\kappa} - R\right) \tag{6}$$

If $R_y r_x < 0$

$$z_{\text{corr}} = z + 2\left(\frac{r_x}{1+\kappa} - R\right) - 2R \tag{7}$$

Range of x & y for Solutions at Real Numbers

In order to avoid complex results for z, the range of the x- and y-coordinates can be limited. Therefore, the terms under the square roots of to equation (2) are considered. For the term within the inner square root it results

$$r_x^2 - (1+\kappa)y^2 \geq 0$$
$$\frac{r_x^2}{(1+\kappa)} \geq y^2$$
$$-\frac{|r_x|}{\sqrt{1+\kappa}} \leq y \leq \frac{|r_x|}{\sqrt{1+\kappa}} \tag{8}$$

If $R_y \neq 0$, the outer square root results with real values for

$$\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - r_x\right)\right)^2 \geq x^2$$

$$\left|R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - r_x\right)\right| \geq |x|$$

A more conservative estimation leads to

$$\left||R_y| + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - |r_x|\right)\right| \geq |x|.$$

Normal Vector of Toroidal Surface

The surface description given by equation (2) can also be interpreted as a zero-isoplane at the potential $V(x, y, z)$, which is also called implicit surface representation

$$V(x,y,z) = R_y - \operatorname{sign}(r_x)\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2} - z = 0 \qquad (9)$$

$$\text{if } |R_y| \geq \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x \geq 0\text{, or } |R_y| \leq \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x \leq 0\,.$$

Thereby, the gradient is equal to the direction n_i of the unit vector, which is orthogonal to the surface.

$$\boldsymbol{n} = \operatorname{grad}\big(V(x,y,z)\big) = \begin{bmatrix} \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{bmatrix}^T$$

At first, the partial derivative in the x-direction leads to

$$\frac{\partial V}{\partial x} = -\frac{1}{2}\operatorname{sign}(r_x)\left(\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2\right)^{-\frac{1}{2}} (-2x)$$

$$= \operatorname{sign}(r_x)\frac{x}{\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}} = z_x\,.$$

Second, the partial derivative in the y-direction leads to

$$\frac{\partial V}{\partial y} = -\frac{1}{2}\operatorname{sign}(r_x)\Bigg(\underbrace{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}_{u(x,y)}\Bigg)^{-\frac{1}{2}} \frac{\partial u_1(x,y)}{\partial y}\,,$$

where

$$\frac{\partial u_1(x,y)}{\partial y} = 2\left(R_y + \frac{1}{1+\kappa}\left(r_x^2-(1+\kappa)y^2\right)^{\frac{1}{2}} - \frac{r_x}{1+\kappa}\right)\frac{\partial u_2(x,y)}{\partial y}$$

and where

$$\frac{\partial u_2(x,y)}{\partial y} = \frac{1}{2(1+\kappa)}\left(r_x^2-(1+\kappa)y^2\right)^{-\frac{1}{2}} \left(-2(1+\kappa)y\right)$$

As a consequence, it follows

$$\frac{\partial V}{\partial y} = \cancel{-}\frac{1}{\cancel{2}}\operatorname{sign}(r_x)\Bigg(\underbrace{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}_{u(x,y)}\Bigg)^{-\frac{1}{2}} \ldots$$

$$\cancel{2}\left(R_y + \frac{1}{1+\kappa}\left(r_x^2-(1+\kappa)y^2\right)^{\frac{1}{2}} - \frac{r_x}{1+\kappa}\right)\frac{1}{\cancel{2(1+\kappa)}}\left(r_x^2-(1+\kappa)y^2\right)^{-\frac{1}{2}} \left(\cancel{-2(1+\kappa)}y\right)$$

$$= \operatorname{sign}(r_x)\frac{y\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)}{\sqrt{r_x^2-(1+\kappa)y^2}\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}} = z_y$$

Third, the partial derivative in the y-direction results with

$$\frac{\partial V}{\partial z} = -1$$

In contrast, the partial derivatives for the cases

$$\text{if } \left|R_y\right| < \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x > 0\text{, or } \left|R_y\right| > \left|\frac{r_x}{1+\kappa}\right| \text{ and } r_x < 0$$

results with

$$\frac{\partial V}{\partial x} = \operatorname{sign}(r_x)\frac{x}{\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(-\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}} = z_x\,.$$

$$\frac{\partial V}{\partial y} = \operatorname{sign}(r_x)\frac{y\left(R_y + \frac{1}{1+\kappa}\left(-\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)}{-\sqrt{r_x^2-(1+\kappa)y^2}\,\sqrt{\left(R_y + \frac{1}{1+\kappa}\left(-\sqrt{r_x^2-(1+\kappa)y^2} - r_x\right)\right)^2 - x^2}} = z_y$$

Finally, the normal unit vector results with

$$\tilde{\boldsymbol{n}} = \frac{1}{\sqrt{1+z_x^2+z_y^2}}\begin{bmatrix}-z_x\\-z_y\\1\end{bmatrix}.$$

Intersection of a Ray and a Torus

The intersection of a ray with a surface can be defined by the line

$$\boldsymbol{r}_i = \boldsymbol{r}_{i-1} + a_i \boldsymbol{d}_{i-1} = \begin{bmatrix} r_{i-1,x} \\ r_{i-1,y} \\ r_{i-1,z} \end{bmatrix} + a_i \begin{bmatrix} d_{i-1,x} \\ d_{i-1,y} \\ d_{i-1,z} \end{bmatrix}, \tag{10}$$

which starts at the former intersection point (or starting point) $\boldsymbol{r}_{i-1}$,by the ray direction $\boldsymbol{d}_{i-1}$and the by distance a_i between the intersection points. During the ray tracing, this unknown distance a_i must be computed.

The toroidal surface according to equation (2) can be rearranged to

$$(z - R_y)^2 = \left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - r_x\right)\right)^2 - x^2$$

and the line equation (10) can be insert, which leads to

$$(r_{i-1,z} + a_i d_{i-1,z} - R_y)^2 = \left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2} - r_x\right)\right)^2 - (r_{i-1,x} + a_i d_{i-1,x})^2$$

$$(r_{i-1,z} - R_y + a_i d_{i-1,z})^2 = \left(R_y + \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2} - r_x\right)\right)^2 - \cdots$$
$$(r_{i-1,x}^2 + 2a_i r_{i-1,x} d_{i-1,x} + a_i^2 d_{i-1,x}^2)$$

$$\left((r_{i-1,z} - R_y)^2 + 2a_i(r_{i-1,z} - R_y)d_{i-1,z} + a_i^2 d_{i-1,z}^2\right)$$
$$= R_y^2 + 2R_y \frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2} - r_x\right)$$
$$+ \frac{1}{(1+\kappa)^2}\left(\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2} - r_x\right)^2 - (r_{i-1,x}^2 + 2a_i r_{i-1,x} d_{i-1,x} + a_i^2 d_{i-1,x}^2)$$

$$\left((r_{i-1,z} - R_y)^2 + 2a_i(r_{i-1,z} - R_y)d_{i-1,z} + a_i^2 d_{i-1,z}^2\right) + (r_{i-1,x}^2 + 2a_i r_{i-1,x} d_{i-1,x} + a_i^2 d_{i-1,x}^2)$$
$$= R_y^2 - 2R_y \frac{1}{1+\kappa} r_x + 2R_y \frac{1}{1+\kappa}\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2}$$
$$+ \frac{1}{(1+\kappa)^2}\left(r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2 - 2r_x\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2} + r_x^2\right)$$

$$(r_{i-1,z}^2 - 2r_{i-1,z}R_y + R_y^2 + 2a_i(r_{i-1,z} - R_y)d_{i-1,z} + a_i^2 d_{i-1,z}^2) + (r_{i-1,x}^2 + 2a_i r_{i-1,x} d_{i-1,x} + a_i^2 d_{i-1,x}^2) - R_y^2$$
$$+ 2R_y \frac{1}{1+\kappa} r_x - \frac{1}{(1+\kappa)^2}\left(r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2 + r_x^2\right)$$
$$= 2\frac{1}{1+\kappa}\left(R_y - \frac{r_x}{1+\kappa}\right)\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + a_i d_{i-1,y})^2}$$

$$a_i^2 d_{i-1,x}^2 + a_i^2 d_{i-1,z}^2 + 2a_i r_{i-1,x} d_{i-1,x} + 2a_i(r_{i-1,z} - R_y)d_{i-1,z} + r_{i-1,x}^2 + r_{i-1,z}^2 - 2r_{i-1,z}R_y + \cancel{R_y^2} - \cancel{R_y^2}$$
$$+ 2R_y \frac{1}{1+\kappa} r_x - \frac{1}{(1+\kappa)^2}\left(2r_x^2 - (1+\kappa)(r_{i-1,y}^2 + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)\right)$$
$$= 2\frac{1}{1+\kappa}\left(R_y - \frac{r_x}{1+\kappa}\right)\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)}$$

$$a_i^2 d_{i-1,x}^2 + a_i^2 d_{i-1,z}^2 + 2a_i r_{i-1,x} d_{i-1,x} + 2a_i (r_{i-1,z} - R_y) d_{i-1,z} + r_{i-1,x}^2 + r_{i-1,z}^2 - 2r_{i-1,z} R_y + 2R_y \frac{1}{1+\kappa} r_x - \frac{2r_x^2}{(1+\kappa)^2} + \frac{r_{i-1,y}^2}{(1+\kappa)} + 2a_i \frac{r_{i-1,y} d_{i-1,y}}{(1+\kappa)} + a_i^2 \frac{d_{i-1,y}^2}{(1+\kappa)} = 2\frac{1}{1+\kappa}\left(R_y - \frac{r_x}{1+\kappa}\right)\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)}$$

$$a_i^2 d_{i-1,x}^2 + a_i^2 \frac{d_{i-1,y}^2}{(1+\kappa)} + a_i^2 d_{i-1,z}^2 + 2a_i r_{i-1,x} d_{i-1,x} + 2a_i \frac{r_{i-1,y} d_{i-1,y}}{(1+\kappa)} + 2a_i (r_{i-1,z} - R_y) d_{i-1,z} + r_{i-1,x}^2 + \frac{r_{i-1,y}^2}{(1+\kappa)} + r_{i-1,z}^2 - 2r_{i-1,z} R_y + 2R_y \frac{1}{1+\kappa} r_x - \frac{2r_x^2}{(1+\kappa)^2} = 2\frac{1}{1+\kappa}\left(R_y - \frac{r_x}{1+\kappa}\right)\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)}$$

$$a_i^2 \frac{(1+\kappa)}{2}\left(d_{i-1,x}^2 + \frac{d_{i-1,y}^2}{(1+\kappa)} + d_{i-1,z}^2\right) + a_i (1+\kappa)\left(r_{i-1,x} d_{i-1,x} + \frac{r_{i-1,y} d_{i-1,y}}{(1+\kappa)} + (r_{i-1,z} - R_y) d_{i-1,z}\right) + \frac{(1+\kappa)}{2}\left(r_{i-1,x}^2 + \frac{r_{i-1,y}^2}{(1+\kappa)} + r_{i-1,z}^2\right) - (1+\kappa)\left(r_{i-1,z} R_y + R_y \frac{r_x}{1+\kappa} - \frac{r_x^2}{(1+\kappa)^2}\right) = \left(R_y - \frac{r_x}{1+\kappa}\right)\sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)}$$

with $\xi = \left(R_y - \frac{r_x}{1+\kappa}\right)^{-1}$ it follows

$$a_i^2 \frac{(1+\kappa)\xi}{2}\left(d_{i-1,x}^2 + \frac{d_{i-1,y}^2}{(1+\kappa)} + d_{i-1,z}^2\right) + a_i (1+\kappa)\xi\left(r_{i-1,x} d_{i-1,x} + \frac{r_{i-1,y} d_{i-1,y}}{(1+\kappa)} + (r_{i-1,z} - R_y) d_{i-1,z}\right) + \frac{(1+\kappa)\xi}{2}\left(r_{i-1,x}^2 + \frac{r_{i-1,y}^2}{(1+\kappa)} + r_{i-1,z}^2\right) - (1+\kappa)\xi\left(r_{i-1,z} R_y + R_y \frac{r_x}{1+\kappa} - \frac{r_x^2}{(1+\kappa)^2}\right) = \sqrt{r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)}$$

$$\left[a_i^2 \frac{(1+\kappa)\xi}{2}\left(d_{i-1,x}^2 + \frac{d_{i-1,y}^2}{(1+\kappa)} + d_{i-1,z}^2\right) + a_i (1+\kappa)\xi\left(r_{i-1,x} d_{i-1,x} + \frac{r_{i-1,y} d_{i-1,y}}{(1+\kappa)} + (r_{i-1,z} - R_y) d_{i-1,z}\right) + \frac{(1+\kappa)\xi}{2}\left(r_{i-1,x}^2 + \frac{r_{i-1,y}^2}{(1+\kappa)} + r_{i-1,z}^2\right) - (1+\kappa)\xi\left(r_{i-1,z} R_y + R_y \frac{r_x}{1+\kappa} - \frac{r_x^2}{(1+\kappa)^2}\right)\right]^2 = r_x^2 - (1+\kappa)(r_{i-1,y} + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2)$$

…

- ➔ This leads to a cubic equation, where the solution can be computed by appropriate algorithms, see also the Attachement. However, it is probably simpler to compute the Ray-ConicTorus intersetions with an iterative zero searching algorithm, e.g. the Newton-Raphson Approach.

Special Case: Cylinder Lens

The toroidal surface according to equations (2) and (3) can also be used to describe a cylinder surface with the long axis in the direction of x. Therefore, the radius of rotation must tend to infinity, $R_y \to \infty$,. As an alternative, once can set

$$x = 0 \text{ and } R_y = 0,$$

which leads to

$$z = -\text{sign}(r_x)\frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - \text{sign}(r_x) r_x\right).$$

For the normal unity vector, it yields

$$\frac{\partial V}{\partial x} = 0 = z_x\,,$$

$$\frac{\partial V}{\partial y} = \text{sign}(r_x)\frac{y}{\sqrt{r_x^2 - (1+\kappa)y^2}} = z_y.$$

Cylinder-Line Intersection

$$z = -\text{sign}(r_x)\frac{1}{1+\kappa}\left(\sqrt{r_x^2 - (1+\kappa)y^2} - \text{sign}(r_x) r_x\right)$$

$$-\text{sign}(r_x)(1+\kappa)z = \sqrt{r_x^2 - (1+\kappa)y^2} - \text{sign}(r_x) r_x$$

For $r_x > 0$ and if the y-values are within the range of equation (8), it results

$$-(1+\kappa)z = \sqrt{r_x^2 - (1+\kappa)y^2} - r_x$$

$$(r_x - (1+\kappa)z)^2 = r_x^2 - (1+\kappa)y^2$$

$$r_x^2 - 2(1+\kappa)z\, r_x + (1+\kappa)^2 z^2 = r_x^2 - (1+\kappa)y^2$$

$$(1+\kappa)^2 z^2 + (1+\kappa)y^2 - 2(1+\kappa)z\, r_x = 0$$

$$(1+\kappa)^2\left(r_{i-1,z} + a_i d_{i-1,z}\right)^2 + (1+\kappa)\left(r_{i-1,y} + a_i d_{i-1,y}\right)^2 - 2(1+\kappa)\left(r_{i-1,z} + a_i d_{i-1,z}\right) r_x = 0$$

$$(1+\kappa)^2\left(r_{i-1,z}^2 + 2a_i r_{i-1,z} d_{i-1,z} + a_i^2 d_{i-1,z}^2\right) + (1+\kappa)\left(r_{i-1,y}^2 + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2\right) - 2(1+\kappa)\left(r_{i-1,z} + a_i d_{i-1,z}\right) r_x = 0$$

$$(1+\kappa)\left(r_{i-1,z}^2 + 2a_i r_{i-1,z} d_{i-1,z} + a_i^2 d_{i-1,z}^2\right) + \left(r_{i-1,y}^2 + 2a_i r_{i-1,y} d_{i-1,y} + a_i^2 d_{i-1,y}^2\right) - 2\left(r_{i-1,z} + a_i d_{i-1,z}\right) r_x = 0$$

$$a_i^2\left((1+\kappa)d_{i-1,z}^2 + d_{i-1,y}^2\right) + 2a_i\left((1+\kappa)r_{i-1,z} d_{i-1,z} + r_{i-1,y} d_{i-1,y} - r_x d_{i-1,z}\right) + (1+\kappa)r_{i-1,z}^2 + r_{i-1,y}^2 - 2r_x r_{i-1,z} = 0$$

⇒ The solution of this quadratic equation

$$H a_i^2 + 2F a_i - G = 0$$

where

$$H = (1+\kappa)d_{i-1,z}^2 + d_{i-1,y}^2$$

$$F = (1+\kappa)r_{i-1,z} d_{i-1,z} + r_{i-1,y} d_{i-1,y} - r_x d_{i-1,z}$$

$$G = -(1+\kappa)r_{i-1,z}^2 - r_{i-1,y}^2 + 2r_x r_{i-1,z}$$

is known as

$$\Rightarrow a_{i,1,2} = -\frac{F}{H} \pm \sqrt{\frac{F^2}{H^2} - \frac{G}{H}} = \frac{\left(\pm\sqrt{\frac{F^2}{H^2} - \frac{G}{H}} - \frac{F}{H}\right)\left(\pm\sqrt{\frac{F^2}{H^2} - \frac{G}{H}} + \frac{F}{H}\right)}{\left(\pm\sqrt{\frac{F^2}{H^2} - \frac{G}{H}} + \frac{F}{H}\right)} = \frac{G}{F \pm \sqrt{F^2 + HG}}.$$

In detail it yields

$$\Rightarrow a_i = \frac{G}{F - \sqrt{F^2 + HG}}, \qquad \text{for } r_x > 0$$

$$\Rightarrow a_i = \frac{G}{F + \sqrt{F^2 + HG}}, \qquad \text{for } r_x < 0$$

Parametrical Toroidal Surface Description

A torus can be defined parametrically by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \left(R_y - r_x + r_x \cos\theta\right)\cos\phi \\ r_x \sin\theta \\ \left(R_y - r_x + r_x \cos\theta\right)\sin\phi \end{bmatrix}$$

In case of $\kappa \neq 0$ it is proposed to use the implicit description for the z-term.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \left(R_y - r_x + r_x \cos\theta\right)\cos\phi \\ \frac{1}{\sqrt{1+\kappa}} r_x \sin\theta \\ z_{\mathrm{corr}} \end{bmatrix}$$

Practical Example with MATLAB

A related MATLAB implementation can be found here:
https://de.mathworks.com/matlabcentral/answers/95230-how-do-i-plot-a-toroid-in-matlab

Attachement

The following screenshots help to understand the tori equations, which are from [Reithmann2009][4].

Page 12

The algebraic function of a torus $T_{r,R}(\mathbf{p},\mathbf{n})$ in general position and orientation is

$$f_T(\mathbf{x}) = \left(\|\mathbf{x}-\mathbf{p}\|^2 + R^2 - r^2\right)^2 - 4R^2\|(\mathbf{x}-\mathbf{p})\times\mathbf{n}\|^2. \tag{2.5}$$

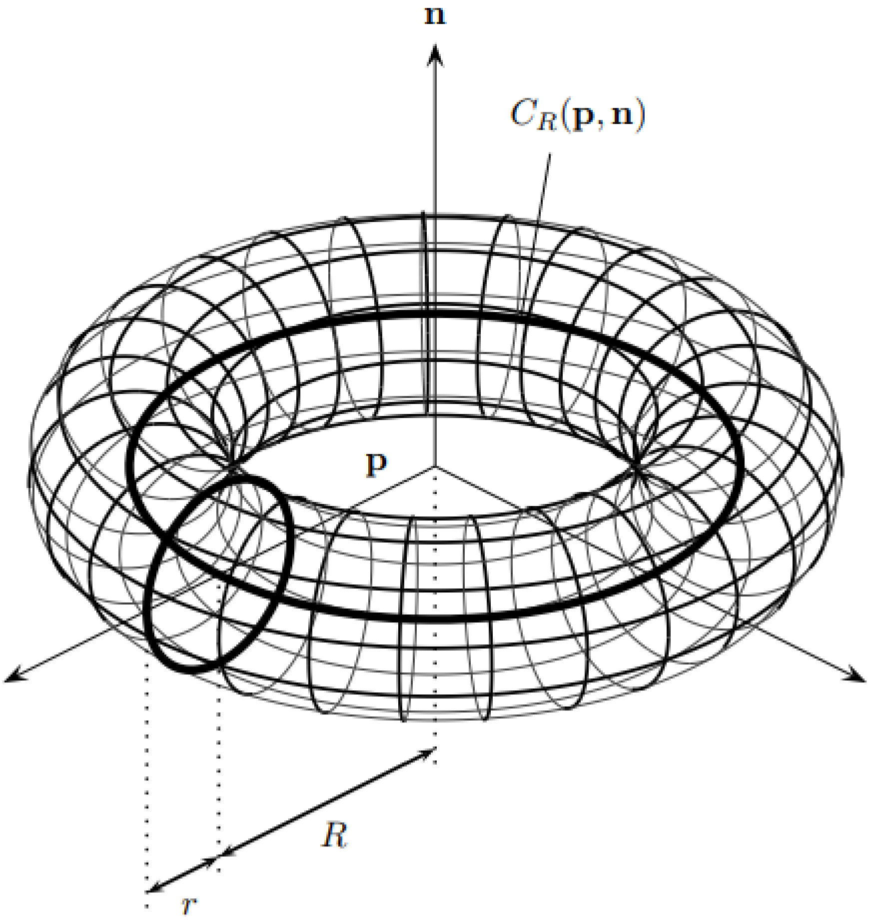

Figure 2.2: Torus $T_{r,R}(\mathbf{p},\mathbf{n})$ with revolving circle

$L(\mathbf{p},\mathbf{d})$	line at point $\mathbf{p}$ and direction $\mathbf{d}$, i.e. $L(\mathbf{p},\mathbf{d}) = \{\mathbf{p}+t\mathbf{d} \mid t \in \mathbb{K}\}$

[4] [Reithmann2009] Reithmann, T.: Topologically correct Intersection Curves of Tori and Natural Quadrics, Dissertation der Johannes Gutenberg-Universität, FB 08: Physik, Mathematik und Informatik, 2009. (See also https://d-nb.info/995451133/34)

4.3.2 Torus-Line Intersection

Given a torus $T = T_{r,R}(\mathbf{o}, \mathbf{e}_3)$ and a line $L = L(\mathbf{p}, \mathbf{d})$. Substituting the parameter form of L into the implicit algebraic equation of T yields

$$f(t) = c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

with

$$\begin{aligned}
c_4 &= \|\mathbf{d}\|^4, \\
c_3 &= 4\|\mathbf{d}\|^2 \langle \mathbf{p}, \mathbf{d} \rangle, \\
c_2 &= 2\|\mathbf{d}\|^2 (\|\mathbf{p}\|^2 + R^2 - r^2) + 4\langle \mathbf{p}, \mathbf{d} \rangle^2 - 4R^2(d_x^2 + d_y^2), \\
c_1 &= 4\langle \mathbf{p}, \mathbf{d} \rangle (\|\mathbf{p}\|^2 + R^2 - r^2) - 8R^2(p_x d_x + p_y d_y), \\
c_0 &= (\|\mathbf{p}\|^2 + R^2 - r^2)^2 - 4R^2(p_x^2 + p_y^2).
\end{aligned}$$

Analogue to previous sections we solve the polynomial and save the intersection points as pairs of a root and a parameter form.

Algorithm 16 `TorusLineIntersection`$(T_{r,R}(\mathbf{o}, \mathbf{e}_3), L(\mathbf{p}, \mathbf{d}))$

Requires: A torus $T_{r,R}(\mathbf{o}, \mathbf{e}_3)$ and a line $L = L(\mathbf{p}, \mathbf{d})$
Returns: A set of regular intersection points $\{\mathbf{q}_r\}$ and a set of singular intersection points $\{\mathbf{q}_s\}$

$a \leftarrow \|\mathbf{p}\|^2$
$b \leftarrow \|\mathbf{d}\|^2$
$c \leftarrow \langle \mathbf{p}, \mathbf{d} \rangle$
$d \leftarrow a + R^2 - r^2$
$c_4 \leftarrow b^2$
$c_3 \leftarrow 4bc$
$c_2 \leftarrow 2bd + 4c^2 - 4R^2(d_x^2 + d_y^2)$
$c_1 \leftarrow 4cd - 8R^2(p_x d_x + p_y d_y)$
$c_0 \leftarrow d^2 - 4R^2(p_x^2 + p_y^2)$
$\{t_s\}, \{t_d\} \leftarrow$ `SolveQuartic`$(c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0)$
for all $t \in \{t_s\}$ **do**
    $\{\mathbf{q}_r\} \leftarrow$ `IntersectionPoint`$(t, \mathfrak{P}_L)$
end for
for all $t \in \{t_d\}$ **do**
    $\{\mathbf{q}_s\} \leftarrow$ `IntersectionPoint`$(t, \mathfrak{P}_L)$
end for
return $\{\mathbf{q}_r\}, \{\mathbf{q}_s\}$